# Discrete Fourier Transform

## 1 DISCRETE FOURIER AND INVERSE FOURIER TRANSFORMS

Compute the discrete Fourier transform $\mathscr{F}\{\mathbf{f}\}$ of the following vectors.

A) $\mathbf{f} = [-1,\ 1]$

B) $\mathbf{f} = [1,\ 1]$

C) $\mathbf{f} = [2,\ 5]$

D) $\mathbf{f} = [0,\ -1,\ 1]$

E) $\mathbf{f} = [2,\ -1,\ -1]$

F) $\mathbf{f} = [2,\ 3,\ 1]$

G) $\mathbf{f} = [-1,\ 0,\ 1,\ 0]$

H) $\mathbf{f} = [-1,\ 1,\ -1,\ 1]$

I) $\mathbf{f} = [3,\ 1,\ -2,\ -4]$

Compute the inverse Fourier transform $\mathscr{F}^{-1}\{\mathbf{c}\}$ of your answers in the previous parts. Verify that you recover $\mathbf{f}$.

## 2 THEORY AND ADVANCED PROBLEMS

A) An **even** vector has the form $\mathbf{f} = [f_0,\ f_1,\ \ldots,\ f_2,\ f_1]$ (i.e $f_{N-k} = f_k$).
An **odd** vector has the form $\mathbf{f} = [f_0,\ f_1,\ \ldots,\ -f_2,\ -f_1]$ (i.e $f_{N-k} = -f_k$).
Show that the discrete Fourier transform of an **even** vector will be pure **real**, and the discrete Fourier transform of an **odd** vector will be pure **imaginary**.

*Hint:* *Use* $\overline{\omega_N}^{N-k} = \overline{\omega_N}^{-k}$ *to show that Fourier transform preserves even / odd, then combine this with* $c_{N-k} = \overline{c_k}$.

B) The discrete Fourier transform $\mathbf{c} = \mathscr{F}\{\mathbf{f}\}$ is often written using matrices as

$$\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \end{bmatrix} = \frac{1}{N} \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0(N-1)} \\ a_{10} & a_{11} & \cdots & a_{1(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(N-1)0} & a_{(N-1)1} & \cdots & a_{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{bmatrix}$$

In this case, the inverse Fourier transform $\mathbf{f} = \mathscr{F}^{-1}\{\mathbf{c}\}$ is written using the conjugate matrix

$$\begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{bmatrix} = \begin{bmatrix} \overline{a_{00}} & \overline{a_{01}} & \cdots & \overline{a_{0(N-1)}} \\ \overline{a_{10}} & \overline{a_{11}} & \cdots & \overline{a_{1(N-1)}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{(N-1)0}} & \overline{a_{(N-1)1}} & \cdots & \overline{a_{(N-1)(N-1)}} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \end{bmatrix}$$

What are the entries of the matrices above? (The matrix for inverse transforms – which is more pretty – is called the “Fourier matrix” $\mathrm{F}_N$).

C) When computing using the Fourier transform, you actually only need to ever calculate the first half of the coefficients, since the second half are conjugate.

Similarly, when computing the inverse fast Fourier transform, you can solve using only the first half of the coefficients. Convert the inverse Fourier formula to only use the first half of the Fourier coefficients.